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ON A PROBLEM OF EVADING MANY PURSUERS

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The motion of one evading point E and n pursuing points in an r-dimensional space is analyzed. The points' velocities change instantaneously and are selected from convex compacta; it is assumed that the set of velocities admissible to point E is wider than that of each of the pursuing points. The nominal motion of point E, i.e., the motion in the absence of pursuers, is a sliding along a specified ray with maximum speed. A piecewise-program control of point E has been constructed, possessing the following property: by staying within a specified neighborhood of the nominal motion, E avoids an explicit contact with all pursuers on an infinite time interval. A lower estimate is given for the maximum distance of point E from all pursuing points. The paper is closely related to [1]. The problem of evading many pursuers was analyzed in [2–5] as well.

1. We consider the motion of one evading point E and n pursuing points $P_1, \ldots, P_n$ in an r-dimensional space $(r \geqslant 2)$

$$x_i^{\cdot} = u_i, \quad u_i \in U_i, \quad i = 1, \ldots, n; \quad y^{\cdot} = v, \quad v \in V \tag{1.1}$$

Here $y, x_1, \ldots, x_n$ are the r-dimensional phase vectors of points $E, P_1, \ldots, P_n$; $V, U_1, \ldots, U_n$ are convex compacta. We assume the fulfilment of the imbeddings

$$U_i \subset \operatorname{int} V, \quad i = 1, \ldots, n \tag{1.2}$$

where $\operatorname{int} V$ denotes the interior of V. The points P_i use arbitrary piecewise-continuous controls with values in U_i, $i = 1, \ldots, n$. At instant t player E has available information on the positions of points $E, P_1, \ldots, P_n$ on the interval $[0, t]$ and on the basis of this information forms his own control at this same instant. The formation of a control v, $v \in V$, such that the realization $v(t)$ is a piecewise-continuous function, is called the strategy of player E. A vector v_0, $v_0 \in \partial V$, where ∂V is the boundary of V, is specified such that the infinite interval θv_0, $\theta > 1$, does not contain the points of V and $\theta v_0 \in \operatorname{int} V$ for some values of θ, $\theta \in [0, 1]$. At the initial instant $t = 0$ point E occupies position E_0, not coinciding with any one of the points $P_1, \ldots, P_n$. The motion of point E from position E_0 with velocity v_0 is said to be nominal.

P r o b l e m. Given a number ε_0, $\varepsilon_0 > 0$. Construct a strategy of player E such that for any admissible controls of the pursuers the point E is found at each instant $t \geqslant 0$ at a positive distance from all the $P_1, \ldots, P_n$, while remaining within the ε_0-neighborhood of the nominal motion. Estimate as well the minimum distance δ_0 from E to the points $P_1, \ldots, P_n$ for $t \geqslant 0$.

A similar problem was first examined by Chernous'ko [1] for the case when V and U_i are spheres with center at the origin, where $U_i \subset V$, $i = 1, \ldots, n$.

2. Without loss of generality we can take it that V does not lie in any hyperplane (otherwise, the problem admits of a lowering of dimension). Hence from (1.2) it follows that a convex polytope Q of full dimensionality can be found such that $U_i \subset Q \subset \operatorname{int} V$, $i = 1, \ldots, n$. Following [6], we shall reckon that the faces and the support hyperplanes of the polytopes are parallel only when their outward normals are parallel.

D e f i n i t i o n 1. A convex polytope is said to be Q-shaped if each face of it corresponds to a parallel face of Q and vice versa.

D e f i n i t i o n 2. A Q-shaped polytope circumscribed around a sphere of radius δ with center at some point A is called the Q_δ-neighborhood $Q_\delta(A)$ of point A. The union of the Q_δ-neighborhoods of the points of a set D is called its Q_δ-neighborhood $Q_\delta(D)$.

We note that for a Q-shaped polytope, a Q-shaped polytope serves as well as its Q_δ-neighborhood, where the distance between the faces equals δ. An $\varepsilon > 0$ can be found such that $U_i \subset Q \subset Q_\varepsilon(Q) \subset V$. If we can solve the problem with U_i replaced by Q, $i = 1, \ldots, n$, and V by $Q_\varepsilon(Q)$, then by the same token the original problem is solved. Therefore, we subsequently take

$$U_i = Q, \quad i = 1, \ldots, n; \quad V = Q_\varepsilon(Q), \quad \varepsilon > 0$$

D e f i n i t i o n 3. Let D_i be Q-shaped polytopes, $i = 1, \ldots, k$. The minimum Q-shaped polytope D containing the D_i is called their Q-union and is denoted $D = [D_1, \ldots, D_k]$.

We consider a point x of the r-dimensional space, whose motion is described by the equation

$$x^{\cdot} = u, \quad u \in Q$$

By $G_t(x(0))$ we denote the attainability domain of x from position $x(0)$ in time t. Let D be some Q-shaped polytope.

$$G_t(D) = \bigcup_{x(0) \in D} G_t(x(0))$$

is called the attainability domain of D. $G_t(D)$ is a Q-shaped polytope. At the instant $t = 0$ let there be specified a ray L with directional vector v_0, having points in common with D, and a point $y(0)$, $y(0) \in \partial D$. We say that from the position $y(0)$, E has a maneuver for escaping from D if a program admissible control $v(t)$ exists such that the corresponding solution $y(t)$ of Eq. (1.1) is found for all $t \geqslant 0$ outside the interior of the attainability domain of D and, beginning from some instant t_*, $y(t) \in L$. Let us indicate the control v permitting the execution of the escape maneuver. We discuss the case $r = 2$ in detail and outline the course of the reasoning for arbitrary r.

We draw the vector v_0 from point zero. The sides of Q and the sides of $G_t(D)$, parallel to them, such that the support lines drawn through them separate polytope Q and the endpoint of vector v_0, are called marked sides. At least one marked side always exists, viz., that which the vector v_0 intersects when leaving Q. To it we assign the number zero. If vector v_0 passes through a vertex of Q, then any one of the sides meeting at this vertex can be taken as the zero side. The segment $L \cap G_t(D)$ has two boundary points. By the point of intersection we shall mean that one of them having the larger coordinate along ray L. Beginning with some instant the point of intersection is located on the zero side of G_t (D). If player E fell into the point of intersection of L with a marked side, then, by moving along L, it can stay outside G_t (D). For this it is sufficient to set $v(t) \equiv v_0$.

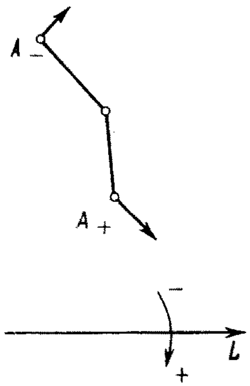

Fig. 1

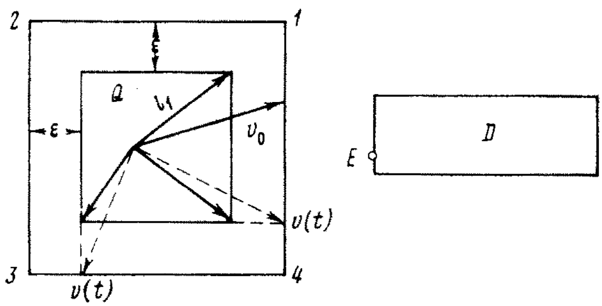

Fig. 2

Let us fix on the direction of positive rotation. Then the half-planes upper and lower with respect to L are defined in a natural manner. It can be shown that the marked sides form a continuous polygonal line and that of the two vertices bounding it one (A_+) moves upward and the other (A_-) downward. If at the initial instant the polygonal line marked on D does not have points in common with L, then we choose a positive direction such that it is found in the lower halfplane (see Fig. 1). Let us number the sides of Q, counting from the zero in the positive direction. From the point zero to the vertices of Q we draw the vectors l_i, $i = 1, \ldots, m$, where m is the number of sides of Q. We number them in succession in the positive direction in such a way that the zero side is included between l_m and l_1. We relate the i-th vertex to the i-th side if $i \neq m$; we relate the m-th vertex to the zero side. The following piecewise-constant control is proposed for player E. If E is located on the i-th side of G_t (D), we set (see Fig. 2)

$$v(t) = l_{i+1} + \frac{l_{i+1} - l_i}{|l_{i+1} - l_i|}\varepsilon, \quad i \neq 0 \qquad (2.1)$$

$$v(t) = l_1 + \frac{l_1 - l_m}{|l_1 - l_m|}\varepsilon, \quad i = 0$$

Under such a choice of control the point E moves along the i-th side of G_t (D), approaching the $(i+1)$st vertex of G_t (D) with velocity ε. Consequently, after a finite time E falls into the $(i+1)$st vertex and, by the same token, onto the

$(i+1)$ st side. The control v is varied in accord with (2.1), and E slides along the $(i+1)$ st side of $G_t(D)$, approaching the $(i+2)$ nd vertex, and so on, until one of two events occurs: either E falls into the point of intersection of L with a marked side or it falls into vertex A_+ before A_+ intersects L. In the first case we set $v(t) = v_0$. In the second, E remains at vertex A_+ until A_+ hits onto L (i.e., vector v equals the corresponding l_i), after which $v(t) = v_0$. We note that under such a choice of control E can intersect L several times.

Let us estimate t_*. Let q and d be the longest sides of Q and D, respectively. We denote t_*^k as the time spent in traversing k sides of $G_t(D)$ during the escape maneuver. Then

$$t_*^1 \leqslant \varepsilon^{-1} d \tag{2.2}$$
$$t_*^i \leqslant t^{i-1} + \varepsilon^{-1}(d + qt_*^{i-1}), \quad i = 2,3, \ldots, m$$

Taking into account that $t_* \leqslant t_*^m$, from the recurrence relations (2.2) we find that $t_* \leqslant [(1+q\varepsilon^{-1})^m - 1]dq^{-1}$. Thus, we have established that a number c_0 can be found such that for any Q-shaped polytope D with largest side no larger than d and for any initial position $E(0) \in \partial D$ the estimate

$$t_* \leqslant c_0 d, \quad c_0 = [(1+q\varepsilon^{-1})^m - 1]q^{-1} \tag{2.3}$$

is valid for the time t_* of the escape maneuver before going onto L. Similarly, a number c_2 can be found such that

$$\operatorname{diam}(D) \leqslant c_2 d \tag{2.4}$$

for every Q-shaped D with largest side no larger than d. As c_2 we can take the integer part of $(m/2)$. If D is circumscribed around a circle of radius δ, then a number c_1 can be found such that

$$d \leqslant c_1 \delta, \quad c_1 = 2 \operatorname{ctg}(\beta/2) \tag{2.5}$$

where β is the smallest angle at a vertex of Q. In this case

$$c_0 d \leqslant c_0 c_1 \delta \quad \text{or} \quad t_* \leqslant c\delta, \ \text{where} \ c = c_0 c_1$$

We now consider the case $r > 2$. The concepts of the zero and the marked sides generalize in a natural way to the case of arbitrary r. It can be shown that the marked faces form a simply-connected set. Beginning with some instant the point of intersection of L with $G_t(D)$ lies on a marked face and, next, on the zero face. Constructing a control of type (2.1), E can fall into any point of that face of $G_t(D)$ on which it is located. At the instant of going onto the boundary of this face a switching takes place, etc. After a finite time t_* player E falls into a point of intersection of L with a marked face, after which $v \equiv v_0$. Inequalities (2.3) –(2.5) remain valid in the general case, if by q and d we mean the longest diameters of the faces of Q and D, respectively.

N o t e. Let two Q-shaped polytopes D and D_1 be specified, where each of them has points not belonging to the other. We consider $D_2(t) = [G_t(D), G_t(D_1)]$. At the initial instant let $E \in \partial D$ and let the goal of player E be to fall onto $\partial D_2(t)$ while remaining outside $G_t(D)$. He can do this by using the maneuver of escaping from D. Indeed, by applying the escape maneuver, E has the capability of falling onto any face of $G_t(D)$. But $\partial D_2(t)$ of necessity contains certain faces of $G_t(D)$. The time of going onto $\partial D(t)$ does not exceed $c_0 d$.

All the subsequent exposition is applicable for arbitrary r, $r \geqslant 2$.

3. Let a decreasing sequence $\Delta : \{\delta_i\}$, $\lim \delta_i = 0$, $i \to \infty$, be specified, all elements of which are positive. The instant that E first falls into the Q_{δ_j}-neighborhood of a certain pursuer is called the instant of j-th encounter with this pursuer (the method for selecting Δ is discussed in Sect. 4). We number the pursuers as follows. Suppose that by instant t we have numbered k of them: $P_1, \ldots, P_k$. We assign the number $k+1$ to that one of the remaining $n-k$ pursuers, with whom the $k+1$-st encounter first occurs (if there are several of them, we select one of them arbitrarily). The instant of the k-th encounter with P_k is denoted t_k. The number δ_1 is selected such that when $t=0$ point E is located outside the interior of the Q_{δ_1}-neighborhood of each pursuer. Note that under such a numbering numbers may not be assigned at all to certain pursuers.

We pass on to describe the strategy of player E. To do this we construct a system $M(t)$ of Q-shaped sets: $M_1(t), \ldots, M_{\alpha(t)}(t)$, where $\alpha(t)$ is the number of sets M_i at instant t. For brevity $M_{\alpha(t)}(t)$ is denoted $M_\alpha(t)$. The motion of player E for $t \geqslant t_1$ reduces to the application of the escape maneuver relative to $M_\alpha(t)$, and, if $\alpha > 1$, E's goal is to go onto the boundary of set $[M_\alpha(t), M_{\alpha-1}(t)]$, while if $\alpha = 1$, it is to go onto L. When $t < t_1$ point E slides along ray L. At the instant t_1 of first encounter $\alpha(t_1) = 1$, $M_1(t_1) = Q_{\delta_1}(P_1(t_1))$. When $t = t_2$ the quantity $\alpha(t_2) = 2$, $M_1(t_2) = G_{t_2 - t_1}(M_1(t_1))$, $M_2(t_2) = Q_{\delta_2}(P_2(t_2))$. Suppose that by instant t we have constructed a system M of Q-shaped sets such that $E(t) \in \partial M_\alpha(t)$. Between switching instants (see below) each of the sets $M_i(t)$ passes in time Δt into $M_i(t+\Delta t) = G_{\Delta t}(M_i(t))$. Point E moves in the escape maneuver mode from $M_\alpha(t)$ up to the switching instant τ when either E goes onto the boundary of $[M_\alpha(\tau), M_{\alpha-1}(\tau)]$, or $\tau = t_{k+1}$, where k is the number of pursuers numbered by the instant $\tau - 0$. In the first case we transform the system $M(\tau - 0)$ into the system $M(\tau + 0)$

$$\begin{aligned} &\alpha(\tau+0) = \alpha(\tau-0) - 1 \\ &M_\alpha(\tau+0) = [M_\alpha(\tau-0), M_{\alpha-1}(\tau-0)] \\ &M_i(\tau+0) = M_i(\tau-0) = G_{\tau-t}(M_i(t)) \\ &i = 1, 2, \ldots, \alpha(\tau-0) - 2 \end{aligned} \tag{3.1}$$

In the second case we transform the system $M(\tau - 0)$ into the system $M(\tau + 0)$

$$\begin{aligned} &\alpha(\tau+0) = \alpha(\tau-0) + 1, \quad M_\alpha(\tau+0) = Q_{\delta_{k+1}}(P_{k+1}(t_{k+1})) \\ &M_i(\tau+0) = M_i(\tau-0) = G_{\tau-t}(M_i(t)), \ i = 1, 2. \ldots, \alpha(\tau-0) \end{aligned} \tag{3.2}$$

The index $\alpha(t)$ varies from 0 to n, and $\alpha(t) = 0$ only when $t < t_1$. We say that by instant t a union of players $P_i, P_{i+1}, \ldots, P_{i+j}$ took place if in the system of $M(t)$ we can find a set $M_s(t)$ containing the attainability domains of the polytopes $Q_{\delta_i}(P_i(t_i)), \ldots, Q_{\delta_{i+j}}(P_{i+j}(t_{i+j}))$. Hence it follows that players $P_i, P_{i+1}, \ldots, P_{i+j}$ remain united on the whole interval $[t, \infty)$ and, furthermore, lose their individuality from the view-point of player E and are replaced by set $M_s(t)$ (the index s depends on t). Beginning with some instant, $\alpha(t)$ becomes equal to unity, and all the pursuers numbered turn out to be united into one set $M_1(t)$. If it happens that $E \in L$ and $\alpha(t) = 1$ on some interval $[\tau, t_{k+1}]$, then for $t \geqslant t_{k+1}$ player E applies the procedure described to the remaining $n-k$ players, counting pursuer P_{k+1} as the first, P_{k+2} as the second, etc., forgetting the existence of $P_1, \ldots, P_k$. The strategy of E, described above, does not, in general, exlude the possibility of coincidence of player E 's position and that of one of the pursuers. If sequence Δ can be selected such that the number of encounters with each pursuer is finite under any actions of the pursuers, then the strategy indicated guarantees E an evasion with a succeeding motion along L.

4. Let us construct sequence Δ. At each instant we separate the system $M(t)$ into two classes. Into the first we place those sets $M_i(t)$ that wholly contain the attainability domain $Q_{\delta_k}(P_k(t_k))$. Into the second we place the rest. If in accordance with the strategy from Sect. 3, E skirts the sets of the first class, then the distance from E to P_k automatically remains not less than δ_k. We define the quantity $\varphi_k^-(n)$ equal to the total time, counting from t_{k+1}, spent by E on the maneuver of escaping from the sets of the second class. Then on the whole half-open interval $[0, \infty)$ the distance from E to P_k will not be less than $\delta_k - \varphi_k^-(n)\, q_0$, where q_0 is the largest velocity of the encounter of E with the pursuer's attainability domain. Note that q_0 is independent of the pursuer's number and that as q_0 we can take $q_0 = \operatorname{diam}(Q) + \varepsilon$. Therefore, if for some positive integer $R(k)$ we have $\delta_k - \varphi_k^-(n)\, q_0 > \delta_{R(k)}$, then the number of encounters of E with P_k does not exceed $R(k)$. Consequently, in order that the number of encounters of E with each pursuer be finite, it is sufficient that for each k, $1 \leqslant k \leqslant n-1$, we find a positive integer $R(k)$, $R(k) > k$, such that the inequality

$$\delta_k - \varphi_k^-(n)\, q_0 > \delta_{R(k)}, \quad 1 \leqslant k \leqslant n-1 \tag{4.1}$$

is fulfilled.

The quantity $\varphi_k^-(n)$ is made up of a finite number of time intervals and depends both on the order of union of the pursuers as well as on the instants at which union takes place. In what follows we shall operate with the quantity $\varphi_k(n)$, being the upper bound of $\varphi_k^-(n)$ and depending only on the order of the union, i.e., on the values $\alpha(t_i)$ and on Δ. The bound on $\varphi_k(n)$ is obtained from $\varphi_k^-(n)$ by making three things more coarse. First, the time of E 's motion along the part of the trajectory of the maneuver of escaping from every Q-shaped set with largest side d is replaced by $c_0 d$ (see (2.3)). Second, if point E went onto $\partial[M_\alpha(t), M_{\alpha-1}(t)]$ at instant t, then in subsequent estimates $[M_\alpha(t), M_{\alpha-1}(t)]$ is replaced by $Q_\gamma(M_{\alpha-1}(t))$, where $\gamma = \operatorname{diam}(M_\alpha(t))$. Third, we make use of the relation $d(\theta + \Delta\theta) = d(\theta) + q\Delta\theta$ to compute the largest side $d(\theta)$ of set $G_\theta(D)$.

We define a quantity $\psi_k(n)$ as equal to $\varphi_k(n)$ if the condition

$$\alpha(t_{k+i} + 0) = \alpha(t_{k+1} + 0) + 1 \quad \text{for all } i, \; n-k \geqslant i > 1 \tag{4.2}$$

is fulfilled. Condition (4.2) signifies that the union of the pursuers takes place in the following sequence: at first P_1 is united with P_2, later P_3 is united with them, next P_4, etc. The quantity $\psi_k(n)$ is found below (see Lemma 1) as an explicit function of Δ. Next, conditions are established (see Lemma 2) for Δ, under which $\psi_k(n) \geqslant \varphi_k(n)$ for all $\alpha(t)$. If these conditions are fulfilled, then we can construct the required sequence Δ by replacing inequalities (4.1) by the stronger

$$\delta_k - \psi_k(n)\, q_0 > \delta_{R(k)}, \quad 1 \leqslant k \leqslant n-1 \tag{4.3}$$

and solving them relative to δ_i.

Lemma 1. The relation

$$\psi_k(n) = c\left\{\delta_{k+1}(1 + \ldots + a^{n-k-1}) + \sum_{i=2}^{n-k} \delta_{k+i}\left(\omega(1 + \ldots + a^{n-(k+i)}) + a^{n+1-(k+i)}\right)\right\}, \quad k = 1, \ldots, n-1; \quad a = 1 + c_0 q; \quad \omega = a c_1 c_2 \tag{4.4}$$

is valid.

Proof. Let $\tau_1^-, \tau_2^-, \ldots$ be the distances between the pursuers at the switching instants, counting from t_{k+1}, and let $\tau_1, \tau_2, \ldots$ be estimates of these distances, derived by the same method as for $\varphi_k(n)$. If condition (4.2) is fulfilled, then

$$t_{k+1} + \tau_1 = t_{k+2}, \quad t_{k+i} + \tau_{2i-2} + \tau_{2i-1} = t_{k+i+1}, \quad i = 2, \ldots, n-1-k$$

Thus, in this case τ_{2i} (τ_{2i+1}) is an estimate of the time τ_{2i}^- (τ_{2i+1}^-) of the maneuver of escaping from P_{k+i+1} (from the union of P_{k+i+1} with $P_{k+1}, \ldots, P_{k+i}$). From the definition of the quantities c_0, c_1, c_2, c follows

$$\begin{aligned} &\tau_1 = c\delta_{k+1}, \quad \tau_{2i-2} = c\delta_{k+i}, \quad \tau_{2i-1} = c_0 d_{2i-2}, \quad i = 2, \ldots, n-k \\ &d_{2i-2} = c_1(\delta_{k+1} + \omega(\delta_{k+2} + \ldots + \delta_{k+i}) + c_0 q(\tau_1 + \ldots + \tau_{2i-2})) \end{aligned} \tag{4.5}$$

where $\omega\delta_{k+i}$ is an estimate of diam $(G_{\tau_{2i-2}}(Q_{\delta_{k+i}}(P_{k+i}(t_{k+i}))))$ and d_{2i-2} is an estimate of the largest side of $M_{\alpha(t)}$ at the instant $t = t_{k+1} + \tau_1^- + \ldots + \tau_{2i-2}^-$. Taking this into account, we can extract expression (4.4) for $\psi_k(n)$ from the recurrence relations (4.5).

The inequality

$$\psi_k(n) > \psi_k(k+j) + \psi_{k+j}(n), \quad 2 \leqslant j \leqslant n-k-1 \tag{4.6}$$

holds. The validity of (4.6) can be established by direct verification, making use of formula (4.4) of the fact that $a > 1$ and that Δ is a decreasing sequence.

Lemma 2. We can select Δ such that

$$\psi_k(n) \geqslant \varphi_k(n) \tag{4.7}$$

under any actions of the pursuers.

The proof is by induction on $n-k$. For $n-k$ equal to one or two, inequality (4.7) can be verified directly for any sequence Δ. Assume that the lemma is valid for $n-k \leqslant f-1$ and prove it for $n-k=f$. We consider two cases. At first we assume that we can find i, $1 \leqslant i \leqslant f$, such that

$$\alpha(t_{k+i} + 0) = \alpha(t_{k+1} + 0) \tag{4.8}$$

This signifies that P_k has been united with P_{k+i-1} and, therefore, $\varphi_k(n)$ can be expanded into the sum $\varphi_k(n) = \varphi_k(k+i-1) + \varphi_{k+i-1}(n)$. The induction hypothesis is valid for each summand; consequently, with due regard to (4.6)

$$\varphi_k(n) \leqslant \psi_k(k+i-1) + \psi_{k+i-1}(n) \leqslant \psi_k(n)$$

Inequality (4.7) is valid for all Δ under assumption (4.8).

We go on to consider the other, more complicated case when

$$\alpha(t_{k+i}+0) > \alpha(t_{k+1}+0), \quad i = 2, \ldots, n-k$$
$$\alpha(t_{k+j}+0) > \alpha(t_{k+1}+0) + 1, \quad \exists j, \; 2 < j \leqslant n-k$$

The index j can always be chosen such that

$$\alpha(t_{k+j}+0) = \alpha(t_{k+j-1}+0) + 1, \quad \alpha(t_{k+j}+0) \geqslant \alpha(t_{k+j+1}+0) \tag{4.9}$$
$$\alpha(t_{k+i}+0) = \alpha(t_{k+1}+0) + 1, \quad i = 1, 2, \ldots, j-1$$

The first inequality in (4.9) signifies that not even one union took place on the interval (t_{k+j-1}, t_{k+j}). The second inequality states that P_{k+j} and P_{k+j-1} were united on the interval (t_{k+j}, t_{k+j+1}).

We introduce into consideration a fictitious auxiliary game (whose elements are distinguished by primes) with $n-1$ pursuers, and a sequence Δ'

$$\delta_i' = \begin{cases} \delta_i, \quad i = 1, \ldots, k+1, \ldots, k+j-2 \\ (\delta_{k+j-1} + \delta_{k+j})a + \delta_{k+j-1} + \omega\delta_{k+j}, \quad i = k+j-1 \\ \delta_{i+1}, \quad i = k+j+1, \ldots, n-1 \end{cases}$$

In the auxiliary game player E' applies the strategy from Sect. 3 to players P_i'; suppose that the latter act such that systems $M(t)$ and $M'(t)$ coincide when $t \leqslant t_{k+j-1}$ and that the relations

$$\alpha'(t_i'+0) = \alpha(t_{i+1}+0); \quad i = k+j, \ldots, n-1$$

are fulfilled, when $t > t_{k+j-1}$. It can be proved that the important inequality

$$\varphi_k'(n-1) > \varphi_k(n) \tag{4.10}$$

is fulfilled. But the induction hypothesis is applicable to the auxiliary game; therefore, from (4.10) follows $\psi_k'(n-1) > \varphi_k(n)$. To complete the proof it remains to find Δ such that $\psi_k'(n-1) \leqslant \psi_k(n)$ when $n-k \geqslant 3$. But for this it is enough to require

$$\delta_{k+1} / \delta_{k+2} \geqslant g'(k+1, n)$$
$$g'(k+1, n) = a^{-1}(1 + \omega(a+1)(a^{n-k-2} - 1)(a-1)^{-1}a^{k+2-n})$$

For the fulfilment of inequalities (4.3) it is sufficient to set

$$\delta_k / \delta_{k+1} \geqslant g''(k, n), \quad k = 1, \ldots, n-1 \tag{4.11}$$
$$g''(k, n) = 1 + cq_0\left\{1 + \frac{1}{a-1}\left(\left(\frac{\omega}{a-1} + 2\right)(a^{n-k} - a) - \omega(n-k-1)\right)\right\}$$

Therefore, if

$$\delta_k / \delta_{k+1} \geqslant g(k, n), \quad k = 2, \ldots, n-1 \tag{4.12}$$

$$g(1, n) = g''(1, n);\quad g(n-1, n) = g''(n-1, n) = 1 + cq_0$$
$$g(k, n) = \max(g'(k, n),\ g''(k, n)),\quad k = 2, 3, \ldots, n-2$$

then relations (4.1) and (4.3) are valid simultaneously. Moreover, (4.11) ensures the fulfilment of (4.3) with $R(k) = k + 1$, i.e.,

$$\delta_k - \psi_k(n)\, q_0 > \delta_{k+1};\quad k = 1, \ldots, n-1, n \tag{4.13}$$

but this signifies that the numbering of the pursuers is not changed if we assign number k to that one of them with whom the k-th encounter first took place. From (4.13) it follows as well that the estimate $\delta_0 > \delta_n$ is valid for the minimum distance δ_0. The quantities $g(k, n)$ in formulas (4.12) depend on Q, ε and $n - k$.

As an example let us derive the values of the parameters in these formulas, when Q is a square with unity side and with center of symmetry at zero. In this case

$$q = 1,\ q_0 = 1 + \varepsilon,\ c_0 = a - 1,\ c_1 = 2,\ c_2 = 2$$
$$c = 2\ (a - 1),\ a = (1 + \varepsilon^{-1})^4,\ \omega = 4a$$

Let us estimate the time T passed by point E outside L. At first we consider the case when the trajectory of E does not contain segments of ray L on which $\alpha(t) = 1$. We define an auxiliary game with $n + 1$ pursuers (its elements are distinguished by double primes) such that $\delta_1'' = \delta_2 g(1, n+1)$, $\delta_{i+1}'' = \delta_i$, $i = 1, 2, \ldots, n$. Thus, the Δ'' defined satisfies (4.12) and, therefore, inequality (4.13) is valid, i.e., $\delta_1'' - q_0\psi_1(n+1) \geqslant \delta''_2$. Consequently,

$$\psi_1(n+1) \leqslant (\delta_1'' - \delta_2'') / q_0 = \delta_1 (g(1, n+1) - 1)\, q_0^{-1}$$

But T does not exceed $\psi_1(n+1)$, therefore

$$T \leqslant \delta_1 (g(1, n+1) - 1) / q_0 \tag{4.14}$$

If E's trajectory contains N segments of L on which $\alpha = 1$, then T can be represented as the sum $T = T_1 + T_2 + \ldots + T_N$, where T_{i+1} is the value of T between the i-th and the $(i + 1)$ st segments. An estimate of type (4.14) is valid for each T_i. We see that it remains valid for their sum. Let us estimate the distance $\rho(t)$ between E and the point accomplishing a nominal motion. We can be convinced that

$$\rho(t) \leqslant q_0 T \leqslant \delta_1 (g(1, n+1) - 1)$$
$$g(1, n+1) = 1 + cq_0 \left\{1 + \frac{1}{a-1}\left(\left(\frac{\omega}{a-1} + 2\right)(a^n - a) - \omega(n-1)\right)\right\}$$

Thus, if we select $\delta_1 \leqslant \varepsilon_0 / (g(1, n+1) - 1)$, then point E remains in the ε_0-neighborhood of the nominal motion.

We note that every system of differential equations $y^\cdot = A(t)\, y + B(t)\, v$, $v \in V_0$, where $A(t)$ and $B(t)$ are matrices of appropriate dimensions, can be brought to the form $\xi^\cdot = v$, $v(t) \in V(t)$. Here ξ is a complete collection of independent first integrals of the system $y^\cdot = A(t)\, y$, while the set of admissible controls $V(t)$ depends upon V_0 and on matrices $A(t)$ and $B(t)$. It can be shown that all the results remain in force for Eqs. (1.1) in which sets U_i and V are time-dependent, if the imbedding

$$\bigcup_{t>0} U_i(t) \subset \operatorname{int} \bigcap^{t>0} V(t); \quad i = 1, \ldots, n$$

are called for instead of (1.2).

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